



# Hierarchical Symbolic Factorization for Sparse Matrices

Aurélien Falco, Emmanuel Agullo, Luc Giraud, Guillaume Sylvand

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# Hierarchical Symbolic Factorization for Sparse Matrices

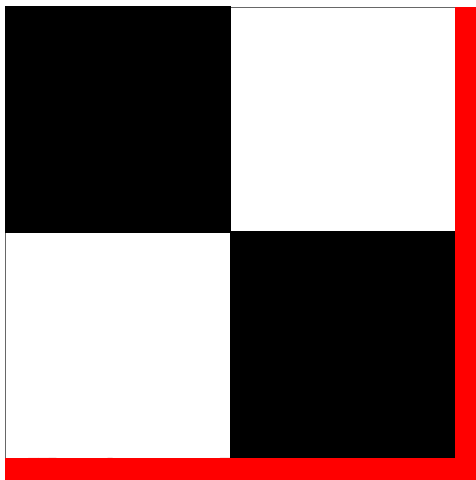
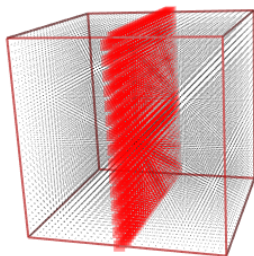
Aurélien FALCO

joint work with:

Emmanuel AGULLO, Luc GIRAUD, Guillaume SYLVAND  
(Airbus & Inria HiePACS team)

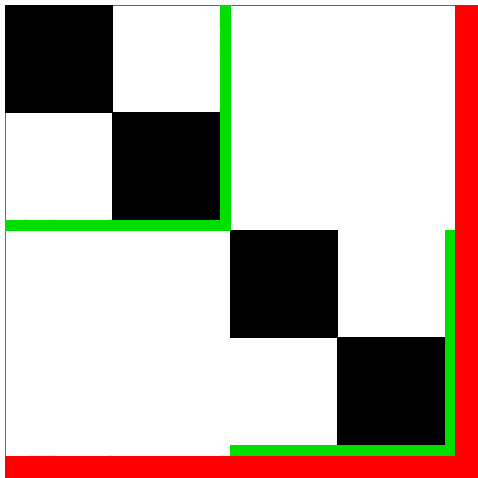
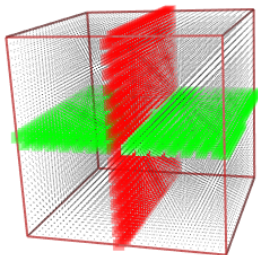
# Sparse direct methods using low-rank compression

Nested dissection (ND) [George, 1973]



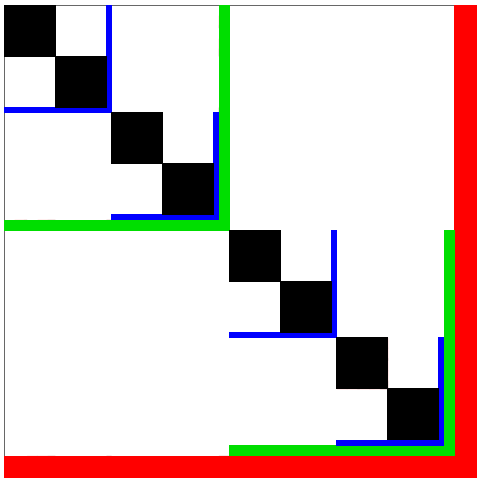
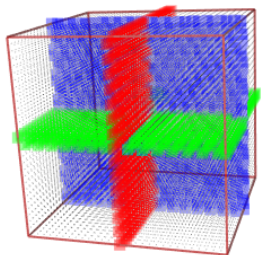
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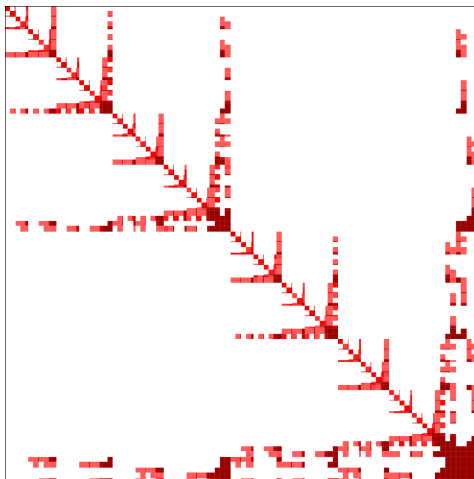
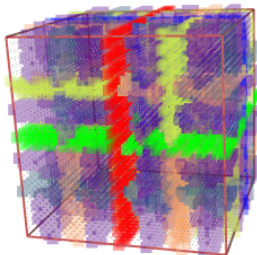
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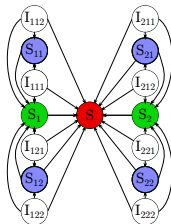
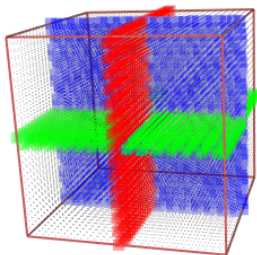
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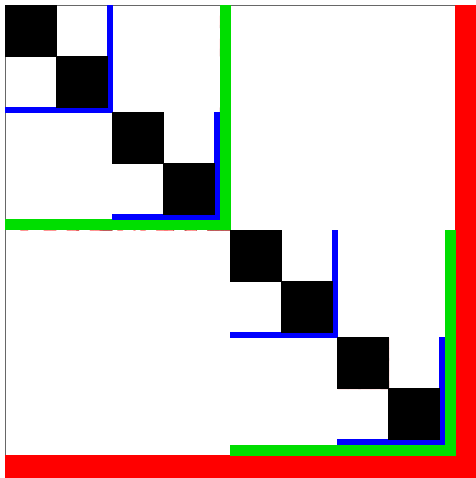


# Sparse direct methods using low-rank compression

## Symbolic Factorization



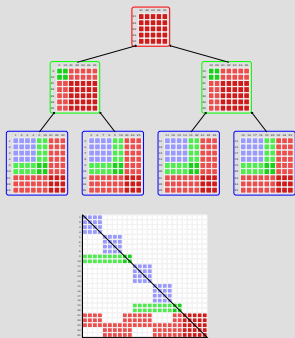
Quotient Graph



Matrix

# Sparse direct methods using low-rank compression

- Multifrontal methods
  - Dense frontal matrices  
[Ghysels et al., 2016, Xia, 2013, Xia et al., 2009, Amestoy et al., 2015, Aminfar et al., 2016, Mary, 2017]
- Supernodal methods
  - Dense (supernodal) sub-matrices  
[Pichon et al., 2017a]



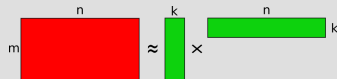
## Sparse methods towards $\mathcal{H}$

- Advantages
  - Orderings
  - Symbolic factorization
- Hierarchical compression used locally on sub-matrices



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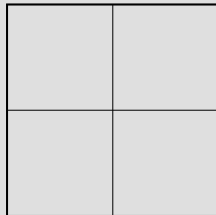


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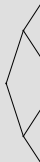
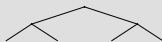
# Hierarchical methods for dense linear systems

- Many Formats:
  - $\mathcal{H}$ -Matrix,  $\mathcal{H}^2$ -matrix, HSS, HODLR, BLR
- Clustering method
  - Bisection  
[Hackbusch, 1999, Hackbusch, 2015]



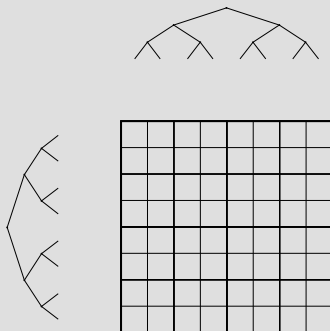
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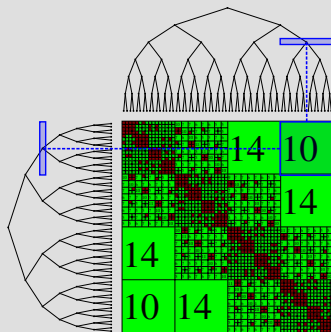

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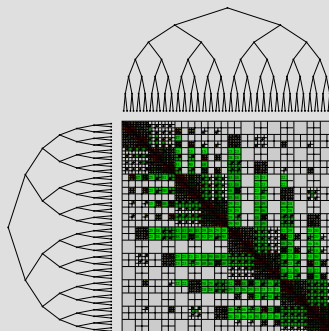
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# Hierarchical methods exploiting sparsity

- Many Formats:
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# Hierarchical methods exploiting sparsity

- Many Formats:
  - $\mathcal{H}$ -Matrix,  $\mathcal{H}^2$ -matrix, HSS, HODLR, BLR
- Clustering method
  - Nested dissection [Le Borne et al., 2007, Grasedyck et al., 2008, Grasedyck et al., 2009, Kriemann, 2013]



## $\mathcal{H}$ -Matrices towards sparse

- Advantages
  - Global hierarchical structure
  - Compression
- How to manage and locate fill-in?

# Problem formulation

## Sparse methods towards $\mathcal{H}$

- Advantages
  - Orderings lead to less fill-in
  - Symbolic factorization
- Current limits?
  - Hierarchical compression used locally on sub-matrices

## $\mathcal{H}$ -Matrices towards sparse

- Advantages
  - Global hierarchical structure
  - Compression
- Current limits?
  - How to manage and locate fill-in?



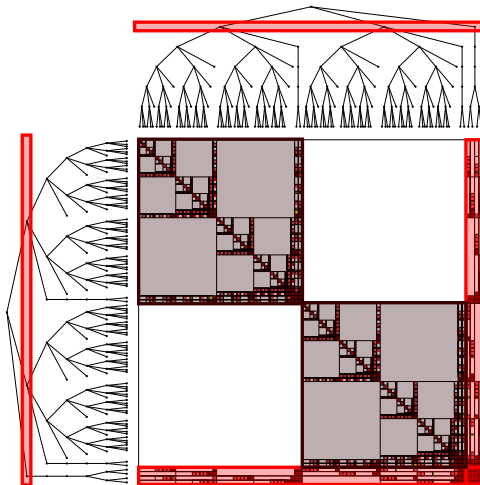
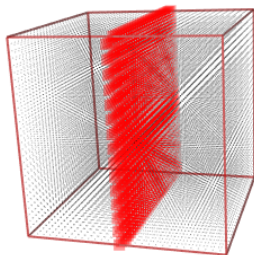
# Objectives

## Combine hierarchical and sparse techniques

- Number of non-zeros as low as sparse direct methods
- A global hierarchical structure

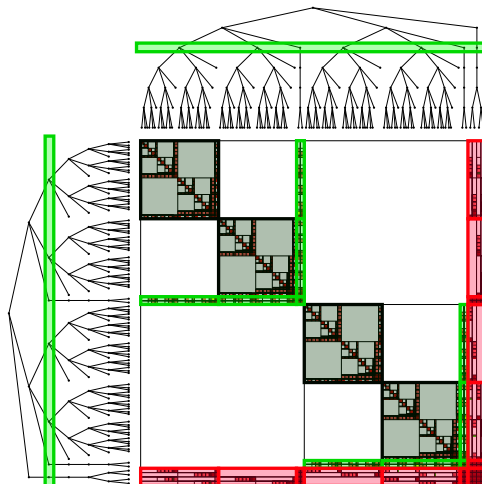
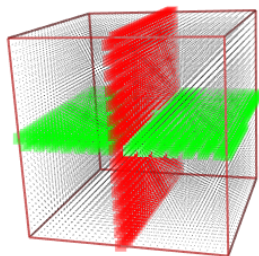
→ Hierarchical Symbolic Factorization

# Nested dissection clustering [George, 1973]



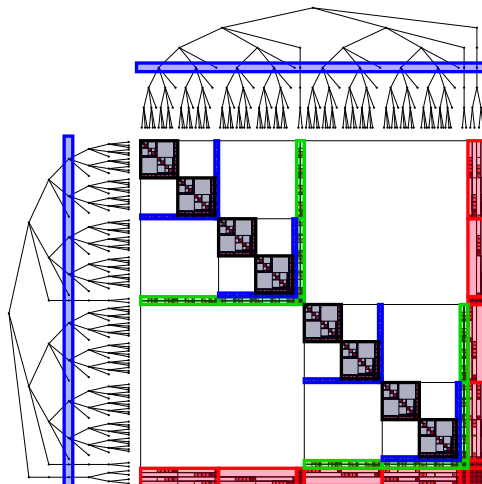
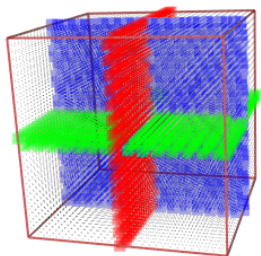
[Le Borne et al., 2007, Grasedyck et al., 2008,  
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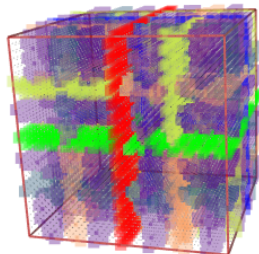
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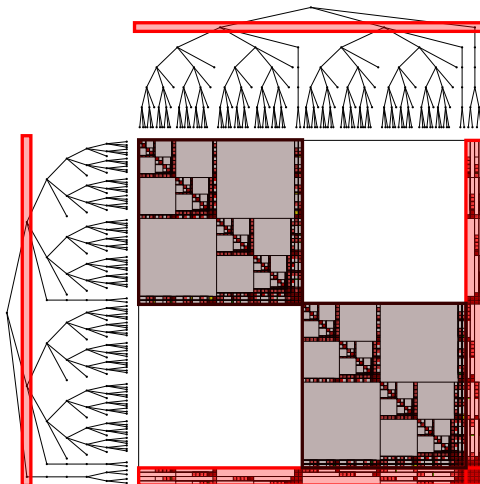
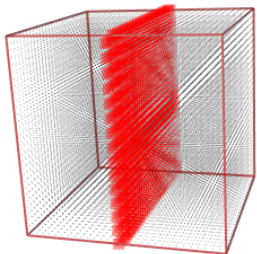
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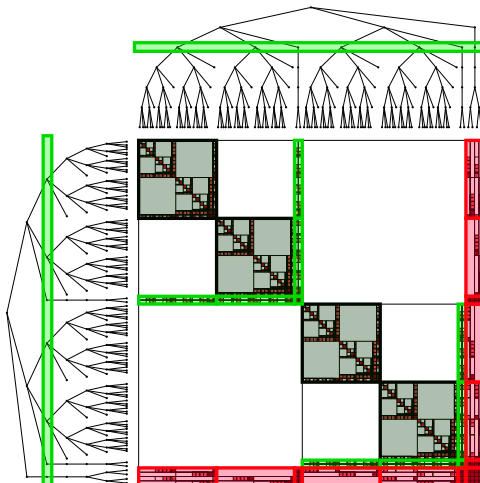
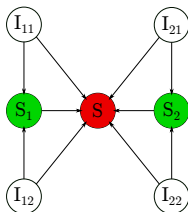
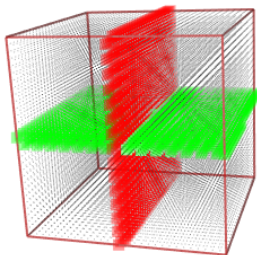
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# Hierarchical Block Symbolic Factorization (TD-HBSF)



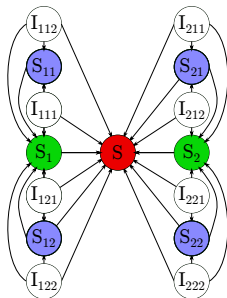
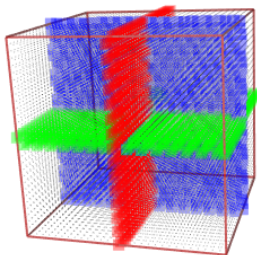
Quotient Graph

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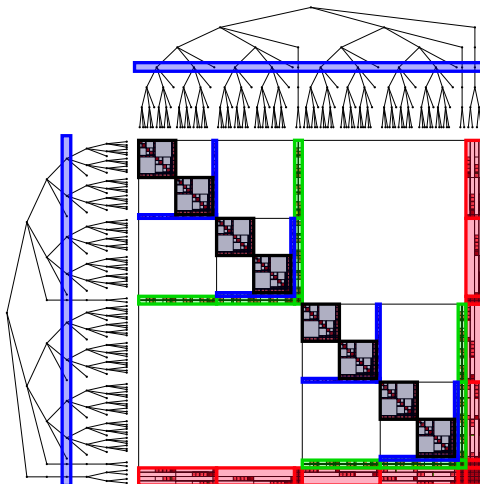


Quotient Graph

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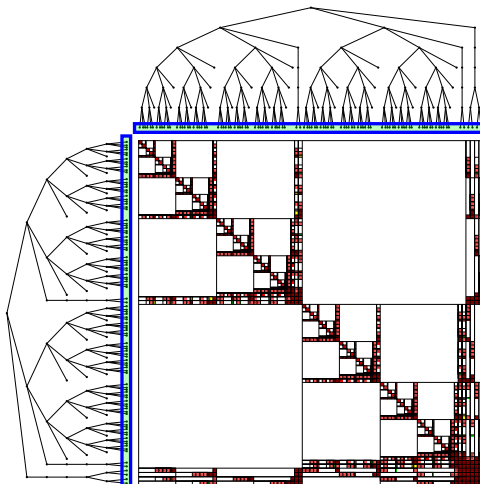
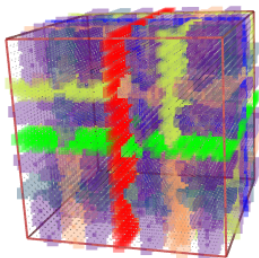


Quotient Graph

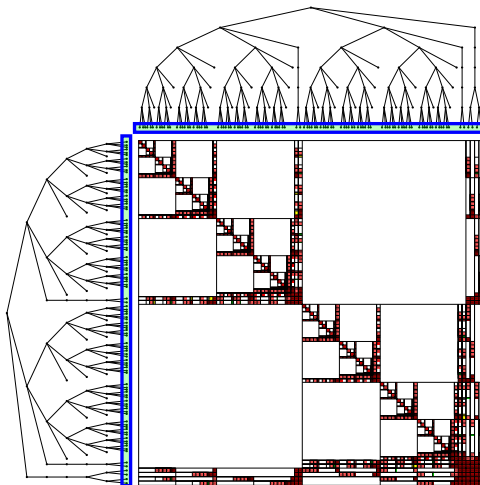
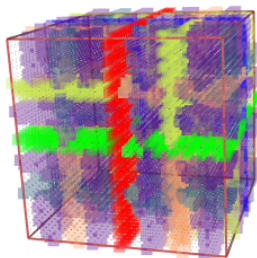




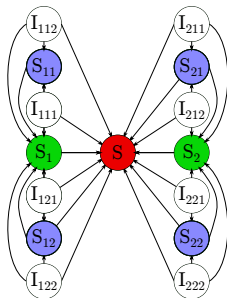
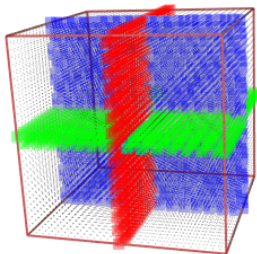
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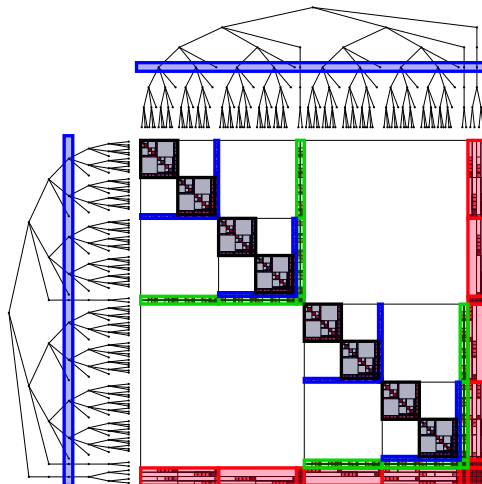
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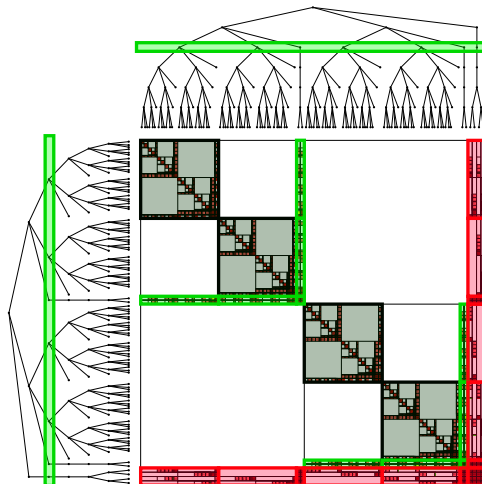
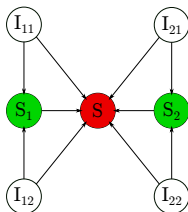
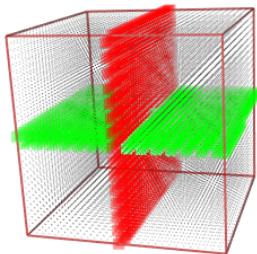
# Hierarchical Block Symbolic Factorization (BU-HBSF)



Quotient Graph

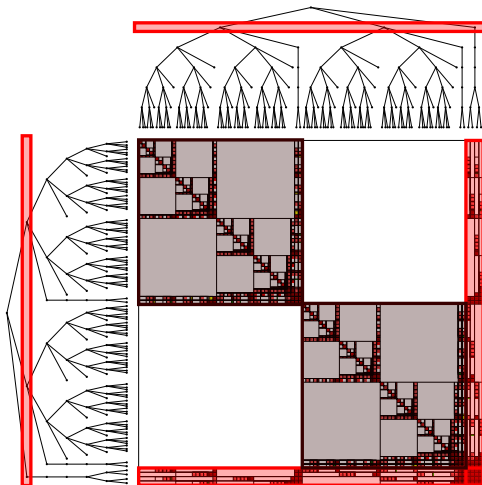
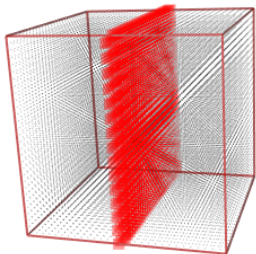


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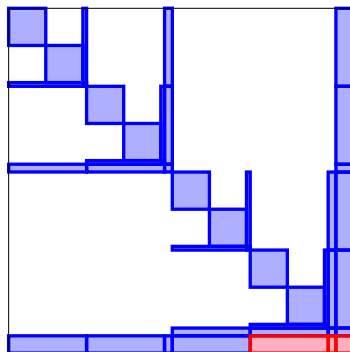
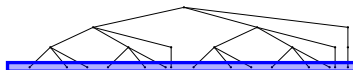
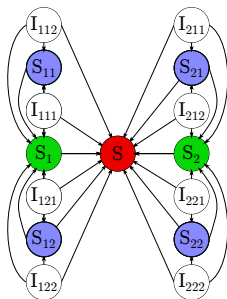
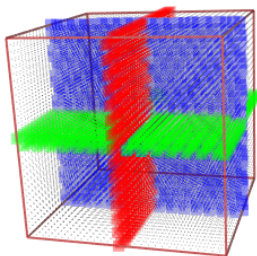
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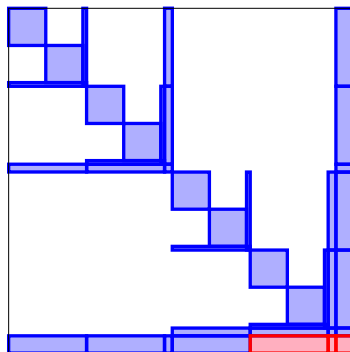
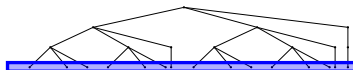
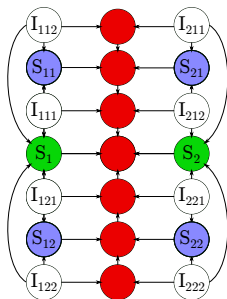
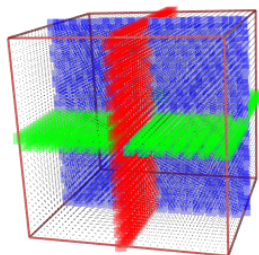


Quotient Graph

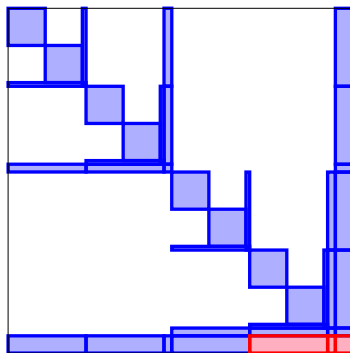
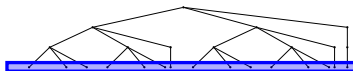
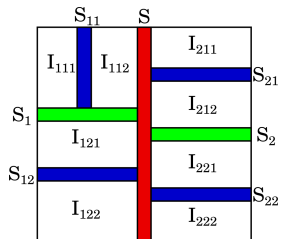
# Separator Division



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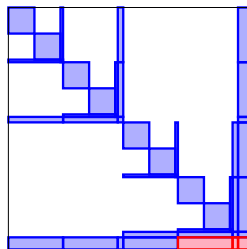
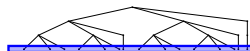
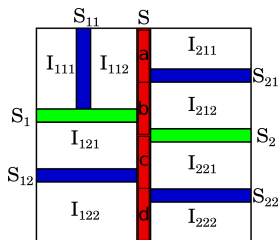
# Separator Division



Minimizing extra-diagonal blocks is NP-hard  
[Pichon et al., 2017b]



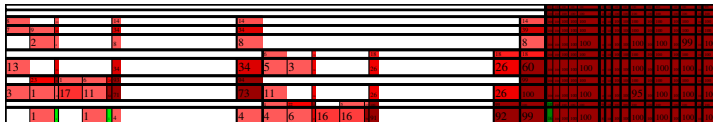
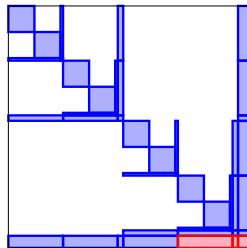
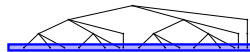
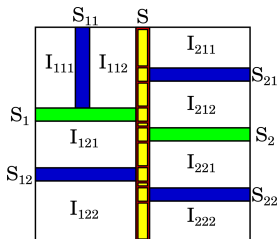
# Interactions-Oblivious Separator Bisection



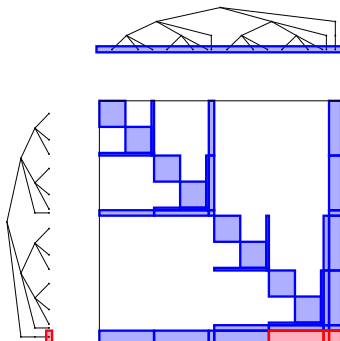
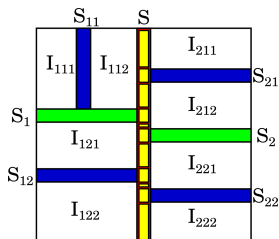
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						2	1			10	10	100	100	100	100	100
11	9	9	8							97	98	100	100	100	100	100
						11	9	9	10		97	100	100	100	98	100

[Le Borne et al., 2007, Grasedyck et al., 2008,  
Grasedyck et al., 2009, Kriemann, 2013]

## Interactions-Aware Flat Separator Division (IA-FSD)

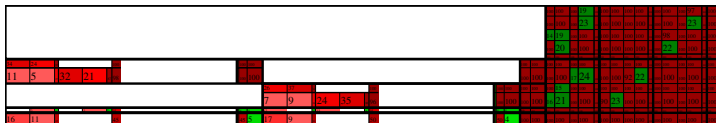
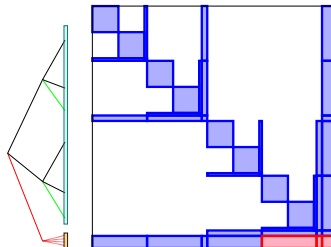
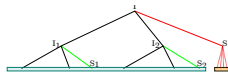
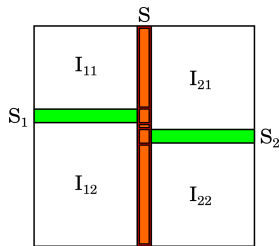


# Interactions-Aware Flat Separator Division (IA-FSD)

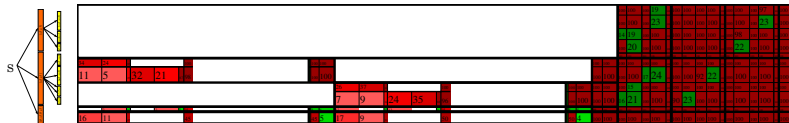
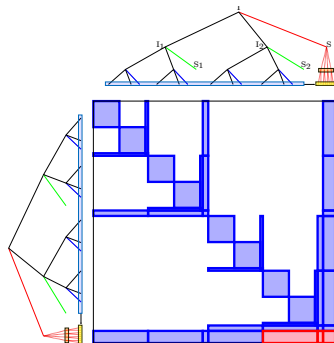
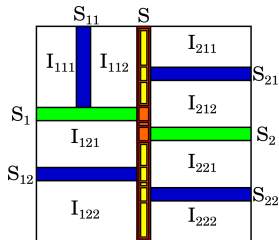


Related to T4 supernodes [Demmel et al., 1999] but too rare and therefore usually rejected

## Interactions-Aware Hierarchical Separator Division (IA-HSD)

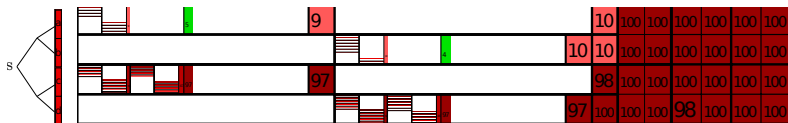
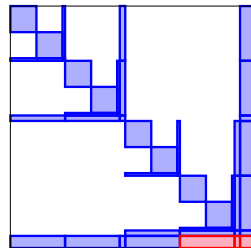
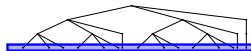
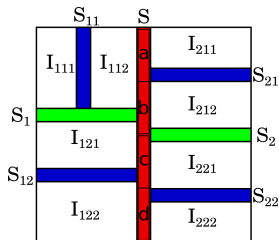


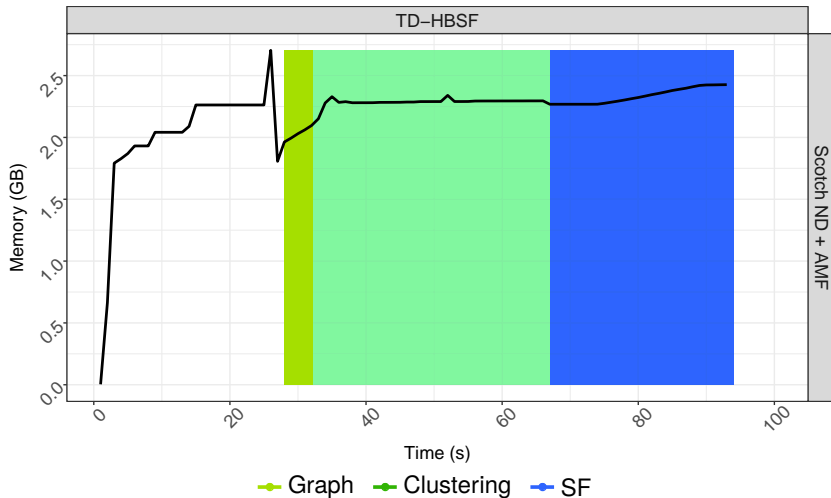
# Interactions-Aware Hierarchical Separator Division (IA-HSD)



# Interactions-Aware Leaf Separator Division (IA-LSD)

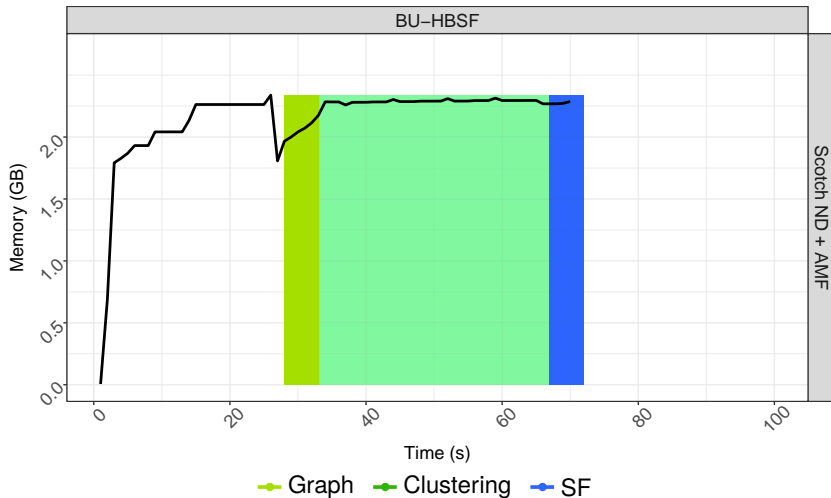
Using Block Column Symbolic Factorization (BCSF) Information





# BU-HBSF Memory Profile

$N = 1e^6$





## Conclusions




- Performing a (bottom-up) hierarchical symbolic factorization is nearly costless
- Possible divisions of separators
  - Bisection
  - Interactions-Aware Flat Separator Division (IA-FSD)
  - Interactions-Aware Hierarchical Separator Division (IA-HSD)
  - Interactions-Aware Leaf Separator Division (IA-LSD)

## Perspectives & Work in Progress



- Extensive comparison between separator division methods
- Study the impact of each algorithm on numerical factorization
- Parallel implementation
- Extensive comparison with reference sparse solvers

Thank you for your attention!




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